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# The rigidly rotating relativistic dust cylinder

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**Abstract.** The solution of van Stockum consists of a rotating dust interior, and three exterior metrics referring to different ranges of the mass per unit length. It has been stated in the literature that the exterior is static, but it is proved here that this is so only in the low-mass case. An examination of the Riemann tensor shows that van Stockum's spacetime is free from singularities and matter at radial infinity below a certain value of the mass per unit length. The ultrarelativistic case, which has closed timelike lines, can occur at physically possible densities and radii.

## 1. Introduction

In a fine paper, well ahead of its time, van Stockum (1937) completely solved the problem of a rigidly rotating infinitely long cylinder of dust, including the application of adequate boundary conditions. The solution is a remarkable one. The metric for the interior is simple and unique, depending on one parameter $a$; but for the vacuum exterior there are three cases, depending on the mass per unit length of the interior. We call these cases the low-mass, null and ultrarelativistic cases, and they will be numbered I, II, III. Of course they satisfy Einstein's vacuum equations and are stationary, i.e. independent of the time coordinate.

It was stated by Frehland (1971) that the exterior metric is static in that it can be diagonalised without introducing the time. This is incorrect: the low-mass case can be diagonalised, but the other two cannot. I prove this here by showing that a hypersurface-orthogonal (HSO) timelike Killing vector exists in case I, but not in cases II or III. The diagonalised form turns out, as expected, to be the Levi Civita metric for a static infinite line-mass. The diagonalisation of case I has also been achieved by Som *et al* (1976).

Even in case I the diagonalisation can be accomplished only by the introduction of a periodic time coordinate, as has been noticed by Tipler (1974a). I shall argue that this spacetime should not be called globally static.

In the ultrarelativistic case the exterior spacetime contains closed timelike lines, and therefore seems to violate causality. The possibility that general relativity permits the existence of time machines (the *construction* of time machines, except in special circumstances, appears to have been ruled out by work of Tipler (1974b)) has been considered by a number of authors recently (Calvani 1978, Charlton 1978, Reboucas 1979). Closed timelike lines have long been known to exist in the Gödel and Kerr metrics, but the physical nature of the sources of these spacetimes is obscure. On the

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other hand, the source of the van Stockum spacetime consists solely of physically reasonable matter, namely dust.

This latter point seemed to me of sufficient importance to merit a careful check. Although it was previously believed (see e.g. Vishveshwara and Winicour 1977) that the van Stockum spacetime arises from nothing but a cylinder of rotating dust, this is not obvious from an inspection of the metrics. I therefore undertake here a study of the Riemann tensor which confirms the non-singular character of the spacetimes, provided the top end of case III is excluded (provided $aR \leqslant 1$: see § 7). I also show that the mass per unit length of cylinder at which the ultrarelativistic regime occurs is by no means impossibly high.

The plan of the paper is as follows. In §§ 2 and 3 the van Stockum solution and some of its properties are given. Section 4 contains the proof of the theorem about the existence of HSO Killing vectors, and in §§ 5 and 6 are given the simplified forms of the exterior metrics in cases I and II when the coordinates are adapted to these vectors. In § 7 (and the Appendix) the Riemann tensor is considered, and it is concluded that, provided $aR < 1$, spacetime is globally regular and free of sources at radial infinity. Mass and angular momentum are considered in § 8, and there is a discussion and conclusion in § 9. The main new work is in §§ 4, 6, 7 and 8.

## 2. The solutions

We use the solutions of van Stockum in his notation, with a few unimportant changes. The metric is

$$ds^2 = -H(dz^2 + dr^2) - L\, d\phi^2 - 2M\, d\phi\, dt + F\, dt^2, \tag{2.1}$$

where $H$, $L$, $M$ and $F$ are functions of $r$ only, and the ranges of the coordinates $z$, $\phi$ and $t$ are

$$-\infty < z < \infty, \qquad 0 \leqslant \phi \leqslant 2\pi, \qquad -\infty < t < \infty, \tag{2.2}$$

the hypersurfaces $\phi = 0$ and $\phi = 2\pi$ being identified. The coordinates will be numbered

$$x^1 \equiv z, \qquad x^2 \equiv r, \qquad x^3 \equiv \phi, \qquad x^4 \equiv t. \tag{2.3}$$

The solution has an interior

$$0 \leqslant r < R \tag{2.4}$$

composed of rotating dust, and a vacuum exterior

$$R < r \tag{2.5}$$

which has three cases according to the magnitude of the mass per unit length (see § 8). The detailed solutions are as follows.

*Interior*, $0 \leqslant r \leqslant R$

$$H = e^{-a^2 r^2}, \qquad L = r^2(1 - a^2 r^2), \qquad M = ar^2, \qquad F = 1, \tag{2.6}$$

where $a$ is an arbitrary positive constant; the density and four-velocity of the dust are given by

$$8\pi\rho = 4a^2\, e^{a^2 r^2}, \qquad u^i = \delta^i_4. \tag{2.7}$$

*Exterior,* $R \leq r$
*Case I,* $aR < \frac{1}{2}$

$$H = e^{-a^2R^2}(R/r)^{2a^2R^2}, \qquad L = \tfrac{1}{2}rR \sinh(3\epsilon + \theta) \operatorname{cosech} 2\epsilon \operatorname{sech} \epsilon,$$
$$M = r \sinh(\epsilon + \theta) \operatorname{cosech} 2\epsilon, \qquad F = rR^{-1} \sinh(\epsilon - \theta) \operatorname{cosech} \epsilon, \tag{2.8}$$

where $\theta = (1 - 4a^2R^2)^{1/2} \log(r/R)$, $\tanh \epsilon = (1 - 4a^2R^2)^{1/2}$.
*Case II,* $aR = \frac{1}{2}$

$$H = e^{-1/4}(R/r)^{1/2}, \qquad L = \tfrac{1}{4}rR[3 + \log(r/R)],$$
$$M = \tfrac{1}{2}r[1 + \log(r/R)], \qquad F = rR^{-1}[1 - \log(r/R)]. \tag{2.9}$$

*Case III,* $aR > \frac{1}{2}$

$$H = e^{-a^2R^2}(R/r)^{2a^2R^2}, \qquad L = \tfrac{1}{2}rR \sin(3\epsilon + \theta) \operatorname{cosec} 2\epsilon \sec \epsilon,$$
$$M = r \sin(\epsilon + \theta) \operatorname{cosec} 2\epsilon, \qquad F = rR^{-1} \sin(\epsilon - \theta) \operatorname{cosec} \epsilon, \tag{2.10}$$

where $\theta = (4a^2R^2 - 1)^{1/2} \log(r/R)$, $\tan \epsilon = (4a^2R^2 - 1)^{1/2} (0 < \epsilon < \frac{1}{2}\pi)$.

The positive square roots are to be taken in all cases. Cases I and III are of Petrov type I, and case II is of Petrov type II (Hoffman 1969). Since in all cases $LF + M^2 = r^2$, the determinant of $g_{ik}$ is $-r^2H^2$, which is non-positive. Hence the metrics have the correct signature for a spacetime.

In the three exterior metrics, $L$, $M$ and $F$ satisfy a linear relation. In case I this is

$$L \cosh^2 \epsilon - RM \cosh 2\epsilon \cosh \epsilon - \tfrac{1}{4}R^2F = 0,$$

and the corresponding relations in the other cases are obtained by replacing $\epsilon$ by $i\epsilon$ (case III) and by zero (case II).

On the boundary $r = R$ the interior and exterior metric coefficients are all continuous, and so are their first derivatives. They therefore satisfy the standard boundary conditions of general relativity.

Singularities will be discussed in § 7. However, we may note here that, if space is to be Euclidean in the neighbourhood of $r = 0$, it is necessary that the ratio of the circumference to the radius of a small circle centred on the axis and in a plane $z =$ constant be $2\pi$. This is so only if the range of $\phi$ is as given in (2.2). Since the ranges of the coordinates $z$, $\phi$, $t$ must be the same on both sides of the boundary $r = R$, the range of $\phi$ in the exterior is also $[0, 2\pi]$.

## 3. Some properties of the solutions

The metrics of § 2 have three and only three linearly independent Killing vectors

$$Z^i = \delta^i_1, \qquad \Phi^i = \delta^i_3, \qquad T^i = \delta^i_4, \tag{3.1}$$

unless $aR = 1$ in exterior case III, when there exists an extra Killing vector

$$Y^i = (z, r, -tR^{-1}, R\phi - t). \tag{3.2}$$

The dust in $r \leq R$ is rigidly rotating, i.e. its shear vanishes. Its angular velocity

$$\omega_{ik} = \tfrac{1}{2}(u_{i;k} - u_{k;i})$$

has for its only non-zero components

$$\omega_{23} = -\omega_{32} = ar,$$

and the vorticity vector $w^i$ given by

$$w^i = \tfrac{1}{2}\eta^{iklm} u_k \omega_{lm}$$

has the single component

$$w^1 = -aH^{-1}$$

and magnitude $|w^i w_i|^{1/2}$ equal to $aH^{-1/2}$. We note that the vorticity scalar depends on $r$, even though the dust is rotating rigidly.

At a point distant $r_1$ from the axis we can make the *purely local* coordinate transformation to the locally non-rotating frame (LNRF) (Bardeen 1970)

$$\mathrm{d}\bar{\phi} = \mathrm{d}\phi + \frac{a}{1-a^2 r_1^2}\,\mathrm{d}t, \qquad \mathrm{d}\bar{t} = \mathrm{d}t, \qquad \mathrm{d}\bar{r} = \mathrm{d}r, \qquad \mathrm{d}\bar{z} = \mathrm{d}z.$$

In this frame the metric has diagonal form near $r_1$. Since the LNRF has angular velocity $-a(1-a^2 r_1^2)^{-1}$ with respect to the co-moving frame of $z$, $r$, $\phi$, $t$, the angular velocity of the fluid with respect to the LNRF is $\omega = a(1-a^2 r_1^2)^{-1}$. As $r_1 \to 0$, $\omega \to a$, so we can take $a$ as the angular velocity on the axis. A similar result was obtained by van Stockum by a different method.

The parameter $a$ is the only one occurring in the interior solution, and both the mass $m$ and the angular momentum $j$ are determined in terms of it and the radius $R$. $m$ and $j$ will be calculated in § 8.

## 4. Existence of timelike hypersurface-orthogonal Killing vectors

The existence of a timelike hypersurface-orthogonal (HSO) Killing vector implies that the metric is static, i.e. that there exists a coordinate system in which the $g_{ik}$ are independent of $t$, and $g_{\alpha 4} = 0 (\alpha = 1, 2, 3)$ (Trautman 1965). Such a coordinate system may not be globally satisfactory: we discuss this in § 5.

*Theorem.* The timelike and null HSO Killing vectors contained in the van Stockum metrics are as follows:

*Interior* $(r < R)$: none.
*Exterior* $(r > R)$ Case I: one timelike;
Case II: one null;
Case III: none.

*Proof.* The most general Killing vector can be written (see § 3)

$$\chi^i = p\delta^i_1 + q\delta^i_3 + s\delta^i_4 + wY^i, \tag{4.1}$$

where $p$, $q$, $s$ and $w$ are real constants, and $w$ is zero unless $aR = 1$ in exterior case III. A necessary condition for this to be HSO is that

$$\chi_{[i,j}\chi_{k]} = 0, \tag{4.2}$$

where the comma means partial differentiation and square brackets mean antisymmetrisation. Taking $i$, $j$, $k$ respectively equal to 1, 2, 3 we at once find $w = 0$ in all cases.

Comparing $i,j,k = 1,2,3$ and $i,j,k = 1,2,4$ we find that either $p = 0$ or

$$\frac{H'}{H} = \frac{qL' + sM'}{qL + sM} = \frac{qM' - sF'}{qM - sF},$$

where the prime means $\mathrm{d}/\mathrm{d}r$, which can be satisfied only if $q = s = 0$. In the latter case (4.1) reduces to $\chi^i = p\delta^i_1$, which is orthogonal to the hypersurfaces $z =$ constant, but spacelike and so of no interest to us here. We therefore take

$$p = 0. \tag{4.3}$$

Putting $i,j,k = 2,3,4$ in (4.2) gives us

$$q^2(LM' - L'M) + qs(L'F - LF') + s^2(FM' - MF') = 0. \tag{4.4}$$

Since $LM' - L'M \neq 0$, $s = 0$, $q \neq 0$ does not lead to an HSO Killing vector, and we may write (4.4) as

$$x^2(LM' - L'M) + x(L'F - LF') + (FM' - MF') = 0, \tag{4.5}$$

where $x = q/s$. The square of the magnitude of the Killing vector (4.1) (with $p = 0$) is

$$g_{ik}\chi^i\chi^k = s^2(F - 2xM - x^2L). \tag{4.6}$$

There exist HSO Killing vectors with $p = 0$ only if the roots of (4.5) are real and independent of $r$; and they are timelike, null or spacelike according to whether (4.6) is positive, zero or negative. We examine the four cases.

*Interior.* (4.5) becomes

$$a^3r^4x^2 + (1 - 2a^2r^2)x + a = 0,$$

both roots of which depend on $r$: hence there are no HSO Killing vectors in this case.

*Exterior case I.* From (4.5) we obtain

$$\tfrac{1}{2}nrR^{-1}\ \mathrm{cosech}\ 2\epsilon\ \mathrm{sech}\ \epsilon(R^2x^2 + 4Rx \cosh 2\epsilon \cosh \epsilon + 4\cosh^2 \epsilon) = 0,$$

where we have put

$$2n = (1 - 4a^2R^2)^{1/2} \tag{4.7}$$

in (2.8), so in this case there exist real roots, independent of $r$, namely

$$x = -2R^{-1}\cosh \epsilon\ \mathrm{e}^{\pm 2\epsilon}. \tag{4.8}$$

The corresponding Killing vectors, obtained from (4.1), (4.3) and $x = q/s$, are

$$\chi^i_1 = s_1(-2R^{-1}\cosh \epsilon\ \mathrm{e}^{2\epsilon}\delta^i_3 + \delta^i_4), \tag{4.9}$$

$$\chi^i_2 = s_2(-2R^{-1}\cosh \epsilon\ \mathrm{e}^{-2\epsilon}\delta^i_3 + \delta^i_4), \tag{4.10}$$

where $s_1$, $s_2$ are arbitrary constants. Their covariant forms are

$$\chi_{1i} = s_1 r\ \mathrm{e}^{\theta+\epsilon}(\mathrm{e}^{2\epsilon}\delta^3_i + 2R^{-1}\cosh \epsilon\ \delta^4_i), \tag{4.11}$$

$$\chi_{2i} = s_2 r\ \mathrm{e}^{-\theta-\epsilon}(\mathrm{e}^{-2\epsilon}\delta^3_i + 2R^{-1}\cosh \epsilon\ \delta^4_i), \tag{4.12}$$

which shows that they are orthogonal to the hypersurfaces

$$f_1(x^a) \equiv \mathrm{e}^{2\epsilon}\phi + 2tR^{-1}\cosh \epsilon = 0, \tag{4.13}$$

$$f_2(x^a) \equiv \mathrm{e}^{-2\epsilon}\phi + 2tR^{-1}\cosh \epsilon = 0 \tag{4.14}$$

respectively. Taking the scalar products of (4.9) with (4.11) and (4.10) with (4.12) we find that $\chi_1^i$ and $\chi_2^i$ are spacelike and timelike respectively. Hence in exterior case I there is one timelike HSO Killing vector.

*Exterior case II.* Equation (4.5) has two coincident roots

$$x = -2R^{-1}.$$

The corresponding single HSO Killing vector is

$$\chi^i = s(-2R^{-1}\delta_3^i + \delta_4^i)$$

which is null and orthogonal to the hypersurface

$$f(x^a) \equiv \tfrac{1}{2}R\phi + t = 0.$$

*Exterior case III.* Equation (4.5) becomes

$$\tfrac{1}{2}n'rR^{-1} \operatorname{cosec} 2\epsilon \sec \epsilon (R^2x^2 + 4Rx \cos 2\epsilon \cos \epsilon + 4\cos^2 \epsilon) = 0,$$

where

$$2n' = +(4a^2R^2 - 1)^{1/2}$$

in (2.10), which has complex roots. Hence the only HSO Killing vector in this case is $\chi^i = p\delta_1^i$, which is spacelike. QED

## 5. Static form of the metric in exterior case I

It was proved by Frehland (1971) and by Som *et al* (1976) that the van Stockum metric can be transformed to static form in exterior case I. The Theorem of the previous section proves also that this *cannot* be done in the other cases. This is a remarkable result, showing that the transformation is possible only if the mass–energy per unit value of $z$ is less than a certain value.

We can obtain the static form of exterior case I by transforming $\phi$ and $t$ to coordinates adapted to the HSO Killing vectors. Put

$$\bar{\phi} = \left(\frac{(2n+1)^3}{16nR^{2n-1}}\right)^{1/2}\left(\phi + \frac{4a}{(2n+1)^2}t\right),$$

$$\bar{t} = \left(\frac{2n+1}{4nR^{1-2n}}\right)^{1/2}\left(t + \frac{(1-2n)^2}{4a}\phi\right), \tag{5.1}$$

where $n$ is given by (4.7), and substitute into (2.1) with $H, L, M, F$ given by (2.8). The result, obtained after a long calculation, is

$$ds^2 = -e^{-a^2R^2}R^{2a^2R^2}r^{2n^2-1/2}(dz^2 + dr^2) - r^{1+2n}\, d\bar{\phi}^2 + r^{1-2n}\, d\bar{t}^2. \tag{5.2}$$

Substituting

$$2C = 1 - 2n \tag{5.3}$$

we obtain

$$ds^2 = -A^2r^{2C^2-2C}(dz^2 + dr^2) - r^{2-2C}\, d\bar{\phi}^2 - r^{2C}\, d\bar{t}^2, \tag{5.4}$$

$A^2$ being written in the place of the constant $e^{-a^2R^2}R^{2a^2R^2}$. *This is the metric for a static line-mass in Weyl coordinates.* It was originally obtained by Levi Civita, and was investigated in detail by Marder (1958). $C$ is related to the mass per unit length, but not exactly equal to it; we shall return to this point in § 8.

It should be noted that, although the exterior can be brought to the static form (5.4), the transformation (5.1) achieves this by introducing a *periodic time coordinate* $\bar{t}$ (Tipler 1974a). To save writing, let us express (5.1) in the form

$$\bar{\phi} = K_1\phi + K_2 t, \qquad \bar{t} = K_3\phi + K_4 t; \tag{5.5}$$

then, since in the original coordinates we identified

$$(t, \phi) \equiv (t, \phi + 2\pi),$$

we must in the new ones make the identification

$$(\bar{t}, \bar{\phi}) \equiv (\bar{t} + 2\pi K_3, \bar{\phi} + 2\pi K_1). \tag{5.6}$$

In the new coordinates there is no smooth match to the interior solution (2.6).

For a spacetime to be globally static it is reasonable to demand the existence of a time coordinate (i) whose level surfaces are orthogonal to a timelike Killing vector, and (ii) which is a monotonically increasing function on future-pointing causal curves. In exterior solution (2.8) there exists a time coordinate (namely $\bar{t}$) satisfying (i), but it does not satisfy (ii). I shall therefore call the solution locally, but not globally, static.

One can verify that the metric of exterior case III ($aR > \frac{1}{2}$) cannot be diagonalised by a real transformation of the form (5.5).

## 6. Null form of the metric in exterior case II

In this case the metric has a null HSO Killing vector, but no timelike one. A linear transformation of $\phi$ and $t$, namely

$$\phi = \tfrac{1}{2}\bar{\phi} + \tfrac{1}{2}R^{-1}\bar{t}, \qquad t = -\tfrac{1}{4}R\bar{\phi} + \tfrac{3}{4}\bar{t},$$

takes the metric into

$$ds^2 = -e^{-1/4}(R/r)^{1/2}(dz^2 + dr^2) - r\, d\bar{\phi}\, d\bar{t} - (r/R)\log(r/R)\, d\bar{t}^2, \tag{6.1}$$

which is a special case of a class of metrics

$$ds^2 = -r^{-1/2}(dz^2 + dr^2) - r\, d\phi\, dt - r\psi\, dt^2,$$

where $\psi$ is a harmonic function, attributed by Kinnersley (1974) to van Stockum. $\bar{t}$ in (6.1) is a null coordinate.

## 7. Singularities and behaviour at infinity

We shall study singularities and behaviour at infinity mainly by means of the Riemann tensor, the components of which for the metric (2.1) are given in the Appendix. The reader is referred to the Appendix for all detailed calculations in this section.

The interior and exterior metrics are differentiable any number of times, and the only possibilities of singularities are at $r = 0$ and $r = \infty$. It is easy to form physical components of the Riemann tensor in the interior case, and these remain finite at $r = 0$ (see Appendix). Moreover, as was remarked at the end of § 2, space is Euclidean at $r = 0$. Hence spacetime is regular throughout the interior.

For the exterior metrics with components (2.8)–(2.10) it is somewhat easier to consider the algebraic invariants of the Riemann tensor. We use four of these given in the Appendix. In all three cases (2.8)–(2.10) of the exterior metric, those invariants which do not vanish are equal to

$$r^{k(a^2R^2-1)} \qquad (k=4 \text{ or } 6), \tag{7.1}$$

but for a constant factor. Hence these invariants are finite for $r>R$ and tend to zero as $r\to\infty$, provided

$$a^2R^2<1. \tag{7.2}$$

If $aR=1$, the invariants are constant. In this case there exists the extra Killing vector (3.2). In fact the spacetime is homogeneous, admitting a four-parameter simply transitive group of motions. It is isometric with a known metric originally discovered by Petrov (1962) and discussed by Debever (1965). The work in this paper provides a possible source for this vacuum metric. This case will be investigated further elsewhere.

If $a^2R^2>1$, the invariants tend to infinity with $r$. Moreover, the proper radial distance from $r=R$ to $r=\infty$ is finite. This suggests that the spacetime has a source at $r=\infty$, in addition to the rotating dust in $r<R$. For this reason we confine attention to (7.2).

If (7.2) is satisfied, the spacetime is globally regular and the invariants of the Riemann tensor tend to zero as $r\to\infty$, suggesting that there are no sources there. Hence we may take the source of this spacetime as an infinitely long rotating cylinder of dust, of everywhere finite density.

## 8. Mass and angular momentum

We take for $m$, the mass per unit $z$ coordinate, and $j$, the angular momentum per unit $z$ coordinate,

$$m=\int \tau^i T^j_i n_j \, \mathrm{d}_3v, \tag{8.1}$$

$$j=-\int \xi^i T^j_i n_j \, \mathrm{d}_3v, \tag{8.2}$$

where $\tau^i$, $\xi^i$ are timelike and rotational Killing vectors respectively, $n_j$ is the unit normal to the spacelike surface of integration, $\mathrm{d}_3v=(-{}^4g)^{1/2}\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\phi$ is the three-dimensional volume element, and the integration is to be carried out over the interior of the cylinder, i.e. over $r<R$ (Hansen and Winicour 1975, Synge 1960).

Because of (4.1), $\tau^i$ and $\xi^i$ need further definition, and we start with the possibilities

$$\tau^i=q\delta^i_3+s\delta^i_4, \tag{8.3}$$

$$\xi^i=q'\delta^i_3+s'\delta^i_4, \tag{8.4}$$

$q$, $s$, $q'$, $s'$ being constants. By demanding that near the axis $r=0$ the trajectories of $\xi^i$ shall be curves of $\phi$ running from 0 to $2\pi$, namely the curves

$$x^3=\phi, \qquad 0\le\phi\le 2\pi, \quad x^i=\text{constant} \quad (i\ne 3),$$

we have $q'=1$, $s'=0$. If we require $\tau^i$ to be a unit vector on $r=0$, then, since

$$|\tau|^2=g_{ik}\tau^i\tau^k=-Lq^2-2Mqs+Fs^2,$$

we have $s^2=1$, because $F=1$ and $L$, $M$ vanish on the axis. To fix $q$, two choices are possible. We can take the trajectories of $\tau^i$ to be those of the dust, in which case $\tau^i=u^i=\delta^i_4$ so $q=0$; or we can use the condition (Vishveshwara and Winicour 1977)

$$\lim_{r\to 0}(\tau^i\xi_i/\xi^i\xi_i)=0,$$

which leads to $q=-a$, and the trajectories of $\xi^i$ for points on the axis are those of locally non-rotating observers (see § 3). We retain $q$ unspecified for the time being and give it the values 0 and $-a$ later. Thus we shall replace (8.3) and (8.4) by

$$\tau^i=q\delta^i_3+\delta^i_4, \tag{8.5}$$

$$\xi^i=\delta^i_3. \tag{8.6}$$

For convenience we choose the hypersurface of integration as $t=$ constant, so that the unit normal is

$$n_i=(1-a^2r^2)^{-1/2}\delta^4_i. \tag{8.7}$$

The contravariant components of $n^i$ are

$$n^i=-a(1-a^2r^2)^{-1/2}\delta^i_3+(1-a^2r^2)^{1/2}\delta^i_4.$$

On the axis, $n^i=\tau^i$ with $q=-a$, which is an argument in favour of this choice of $q$.

Using

$$T^j_i=\rho u^j u_i=\rho\delta^j_4(-M\delta^3_i+\delta^4_i)$$

and (8.5), (8.6) and (8.7) we can now evaluate (8.1) and (8.2). We find

$$m=(1-\tfrac{2}{3}q/a)[1-(1-a^2R^2)^{1/2}]+\tfrac{1}{3}qaR^2(1-a^2R^2)^{1/2}, \tag{8.8}$$

$$j=\tfrac{2}{3}a^{-1}[1-(1-a^2R^2)^{1/2}]-\tfrac{1}{3}aR^2(1-a^2R^2)^{1/2}. \tag{8.9}$$

If $a^2R^2\ll 1$, so that terms of order $a^4R^4$ in $m$ and $a^5R^6$ in $j$ may be neglected, these reduce to

$$m=\tfrac{1}{2}a^2R^2, \tag{8.10}$$

$$j=\tfrac{1}{4}a^3R^4, \tag{8.11}$$

which are precisely the Newtonian values for the mass and angular momentum per unit length of a rigidly rotating cylinder of dust with angular velocity $a$ and radius $R$, units being chosen so that $G=1$. We notice that $q$ does not enter the expression (8.9) for the angular momentum, but does occur in the mass (8.8). Putting $q=-a$ for the reason explained, we find that its presence makes little difference to the mass until $aR$ gets large: for example, at the borderline between cases I and II, when $aR=\tfrac{1}{2}$, the masses per unit $z$ for $q=0$ and $q=-a$ are about 0·13 and 0·15 relativistic units respectively; but at the extreme limit of physical significance of case III, namely $aR=1$, we find that the corresponding figures are 1 and $\tfrac{5}{3}$.

In § 5 the exterior case I metric was reduced to the static form (5.2) which was similar to the Levi Civita form (5.4). The constant $C$ in the latter is approximately twice the gravitational mass per unit $z$ (Marder 1958) so we have, using (4.7),

$$m\sim\tfrac{1}{2}C=\tfrac{1}{4}(1-2n)\sim\tfrac{1}{2}a^2R^2,$$

agreeing with (8.10).

We notice that $aR$, the value of which determines the case the metric falls into, is related to the mass rather than the angular momentum. In fact it is possible to have a metric falling in the ultrarelativistic case III ($aR > \frac{1}{2}$) and yet have the angular momentum small (if $a$ is large and $R$ small).

Introducing units of customary dimensions we may write (8.10) as

$$Gm/c^2 = \tfrac{1}{2}a^2R^2/c^2.$$

Case III ($aR/c > \frac{1}{2}$) requires

$$Gm/c^2 > \tfrac{1}{8},$$

$m$ being an approximation to the mass per unit length. It follows from this, as recognised by Vishveshwara and Winicour (1977), that the ultrarelativistic case could occur when the radius and density of the infinite cylinder were about the same as those at which horizons form in finite bodies.

We summarise this section. The mass and angular momentum are given by (8.8) and (8.9). More than one definition of mass is possible, but for $aR \leqslant \frac{1}{2}$ (8.10) applies with good approximation. The ultrarelativistic case $aR > \frac{1}{2}$ does not require high densities or angular momenta.

## 9. Discussion and conclusion

It has long been realised that van Stockum's spacetime can contain closed timelike lines. These occur in the exterior case III metric (2.10) where $L$ is negative because the curve

$$x^3 = \phi, \qquad x^i = \text{constant} \quad (i \neq 3),$$

becomes timelike, and, assuming that we preserve (2.2), it is closed. Closed timelike lines are known in other relativistic metrics, e.g. the Gödel and Kerr metrics, but one does not know in those cases whether the sources are made out of reasonable matter. What seems now quite clear from §§ 7 and 8 of this paper is that closed timelike lines can occur in a globally regular metric whose source consists of realistic matter (i.e. dust).

The remaining doubt, of course, is whether a cylinder of infinite length is plausible enough for physical conclusions to be drawn from a study of it. Undoubtedly the infinite length of the cylinder causes some strange effects, rotating or not. For example, though the spacetime tends to flatness at infinity, it is not globally Euclidean because the ratio of the circumference to the radius of large circles centred on the axis is not $2\pi$. Further, test particles cannot escape from the gravitational field of the rod to infinity: this happens even with Newtonian infinite rods. Nevertheless, in some respects an infinite cylinder may be a model for a long finite one, and the possibility cannot be dismissed that a time machine might be associated with a long, but finite rotating system.

Another result proved in this paper is that exterior cases II and III are not static. Case I is static in the sense that the metric can be diagonalised, but only at the cost of introducing a periodic time coordinate.

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**Appendix. The Riemann tensor**

The non-zero components of $R_{ijkl}$ for the metric (2.1) can be expressed in terms of the following.

$$R_{1212}=\tfrac{1}{2}H''-\tfrac{1}{2}H^{-1}H'^2, \qquad R_{1313}=\tfrac{1}{4}H^{-1}H'L', \qquad R_{1414}=-\tfrac{1}{4}H^{-1}F'H',$$
$$R_{2323}=\tfrac{1}{2}L''-\tfrac{1}{4}H^{-1}H'L'-\tfrac{1}{2}r^{-1}L'+\tfrac{1}{4}r^{-2}L(F'L'+M'^2),$$
$$R_{2424}=-\tfrac{1}{2}F''+\tfrac{1}{4}H^{-1}F'H'+\tfrac{1}{2}r^{-1}F'-\tfrac{1}{4}r^{-2}F(F'L'+M'^2), \tag{A1}$$
$$R_{3434}=-\tfrac{1}{4}H^{-1}(F'L'+M'^2), \qquad R_{1314}=\tfrac{1}{4}H^{-1}H'M',$$
$$R_{2324}=\tfrac{1}{2}M''-\tfrac{1}{4}H^{-1}H'M'-\tfrac{1}{2}r^{-1}M'+\tfrac{1}{4}r^{-2}M(F'L'+M'^2).$$

The dual Riemann tensor

$$R^{*ij}{}_{kl}=\tfrac{1}{2}\eta^{ijmn}R_{mnkl}$$

has non-zero components expressible in terms of $R^{*12}{}_{34}$, $R^{*13}{}_{23}$, $R^{*13}{}_{24}$, $R^{*14}{}_{23}$, $R^{*14}{}_{24}$, $R^{*23}{}_{13}$, $R^{*23}{}_{14}$, $R^{*24}{}_{13}$, $R^{*24}{}_{14}$, which are easily obtained from (A1).

For the interior $r<R$ we can form the physical components of the Riemann tensor as follows. We choose differential forms

$$\theta^1=H^{1/2}\,dz, \qquad \theta^2=H^{1/2}\,dr, \qquad \theta^3=r\,d\phi, \qquad \theta^4=-ar^2\,d\phi+dt,$$

with the corresponding basis vectors

$$e_1^i=H^{-1/2}\delta_1^i, \qquad e_2^i=H^{-1/2}\delta_2^i, \qquad e_3^i=r^{-1}\delta_3^i+ar\delta_4^i, \qquad e_4^i=\delta_4^i, \tag{A2}$$

so that metric (2.1) becomes

$$ds^2=-(\theta^1)^2-(\theta^2)^2-(\theta^3)^2+(\theta^4)^2,$$

and the physical or tetrad components of the Riemann tensor are

$$R_{abcd}=e^i{}_a e^j{}_b e^k{}_c e^l{}_d R_{ijkl},$$

$R_{ijkl}$ denoting coordinate components. The latter are all finite at $r=0$, as can be seen from (2.6) and (A1). From (A2) it is evident that the only danger of a singularity at $r=0$ comes from components in which one or two of $a$, $b$, $c$, $d$ are equal to 3. The situation is saved because components of $R_{ijkl}$ with one or two 3's in them contain a factor $r^2$. Thus all components of $R_{abcd}$ are finite at $r=0$.

We turn now to the vacuum exterior, $r>R$. Algebraic invariants of the Riemann tensor for vacuum spacetimes are (Weinberg 1972)

$$R_{ijkl}R^{ijkl}, \tag{A3}$$
$$R^{*ij}{}_{kl}R^{kl}{}_{ij}, \tag{A4}$$
$$R_{ijkl}R^{klmn}R_{mn}{}^{ij}, \tag{A5}$$
$$R^{*ij}{}_{kl}R^{klmn}R_{mnij}. \tag{A6}$$

It turns out that for *all three* exterior cases (2.8)–(2.10) the mixed components of the Riemann tensor may be written in terms of

$$R^{12}{}_{12}=Py^h, \qquad R^{13}{}_{13}=-P(1-a^2R^2)y^h, \qquad R^{13}{}_{14}=-aPy^h,$$
$$R^{14}{}_{14}=-a^2R^2Py^h, \qquad R^{23}{}_{23}=-a^2R^2Py^h, \qquad R^{23}{}_{24}=aPy^h,$$

$$R^{24}{}_{24} = -(1 - a^2R^2)Py^h, \qquad R^{34}{}_{34} = Py^h,$$

where, for brevity, we have written $P = a^2 e^{a^2R^2}$, $y = r/R$, $h = 2a^2R^2 - 2$. It is now easy to check that invariants (A3) and (A5) have the form (7.1). The invariants (A4) and (A6) are easily seen to vanish identically.

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